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Integrals of Set-Valued Functions*

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INTRODUCTION

Set-valued functions have been of interest for some time. Fixed-point theorems for such functions were proved by Kakutani [1], Eilenberg and Montgomery [2], and others; furthermore, set-valued functions have been used repeatedly in Economics (see for example Arrow and Debreu [3], McKenzie [4] and Vind [5]). Integrals of set-valued functions have been studied in connection with statistical problems; see Kudo [6] and Richter [7]. Lately integrals of set-valued functions have arisen in connection with economic problems [8, 9], and we here extend the basic theory of such integrals.

Let T be the unit interval [0, 1]. For each t in T, let $F(t)$ be a nonempty subset of euclidean n-space E^n. Let $\mathscr{F}$ be the set of all point-valued functions f from T to E^n such that f is integrable over T and $f(t) \in F(t)$ for all t in T. Define

$$\int_T F(t)\, dt = \left\{ \int_T f(t)\, dt : f \in \mathscr{F} \right\}$$

i.e., the set of all integrals of members of $\mathscr{F}$. This notion is a natural generalization of the integral of point-valued functions on the one hand, and of the sum of a finite number of sets on the other hand. It is closely connected with Vind's "set valued measures" [5]; if for measurable subsets S of T, we set $\nu(S) = \int_S F(t)\, dt$, then ν is a set-valued measure.

The following conventions will be used: Instead of $\int_T F(t)\, dt$, $\int_T f(t)\, dt$, etc., we will write $\int F$, $\int f$, etc.; when it is necessary to integrate over a subset S of T we will write $\int_S F$, etc., but otherwise the range of integration will not be specified and will be understood to be all of T. When we refer to "all" or "each" t in T we will mean "almost all," and when we say that "there is a t in T" with some property, then we shall mean that the set of

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points t in T with the property in question is of positive Lebesgue measure. "Measurable" will mean "Lebesgue measurable". Coordinates of points in E^n will be denoted by superscripts. If x and y are in E^n, $x \geqslant y$ will mean[1] $x^i \geqslant y^i$ for all i. The scalar product is denoted $x \cdot y$, and $| x |$ means $(| x^1 |, \cdots, | x^n |)$. Set-theoretic subtraction will be denoted $\backslash$.

The following basic theorem is due to Richter [7]:

THEOREM 1. *$\int F$ is convex.*

It is natural to ask under what conditions $\int F$ (or equivalently, $\mathscr{F}$) is nonempty. The function F will be called *Borel-measurable* if its graph $\{(t, x) : x \in F(t)\}$ is a Borel subset[1a] of $T \times E^n$. It will be called *integrably bounded* if there is a point-valued integrable function h from T to E^n such that $| x | \leqslant h(t)$ for all x and t such that $x \in F(t)$.

THEOREM 2. *If F is Borel-measurable and integrably bounded, then $\int F$ is nonempty.*

Neither hypothesis in this theorem can be omitted, as we shall show by counter-example. This theorem follows immediately from a basic "measurable choice theorem" due to von Neumann [11].

The function F will be called *nonnegative* if $x \geqslant 0$ for all x and t such that $x \in F(t)$. For each t in T, $F^*(t)$ will denote the convex hull of $F(t)$.

THEOREM 3. *If F is nonnegative and Borel-measurable, then $\int F = \int F^*$.*

This theorem looks like a trivial consequence of Theorem 1, but it isn't. Neither hypothesis can be removed. Note that the theorem remains true if F is bounded from below (or from above) by an integrable point-valued function h, so that in effect the nonnegativity condition is a weakening of the condition of integrable boundedness.

The function F will be called *closed* if $F(t)$ is closed for each t.

THEOREM 4. *If F is closed and integrably bounded, then $\int F$ is compact.*

Note that the hypothesis of Borel-measurability is not needed here. Theorem 4 was proved by Kudo [6] for F that are convex-valued and Borel measurable (in addition to the conditions given here). Richter [7] proved a version slightly different from Kudo's, which does not require convexity, but which still requires measurability.

[1] Note that this differs from the standard usage, in which this relation is denoted $\geqq$.

[1a] This definition is justified by the fact that a point-valued function is Borel-measurable if and only if its graph is a Borel set (see [10], p. 365, Proposition 4, and p. 398, Proposition 2). For compact convex valued functions, this definition coincides with that of Kudo [6] and Richter [7], who use the measurability of the support function.

We now turn to a generalization of Lebesgue's dominated convergence theorem; this may be considered our chief result. If $A_1, A_2, \cdots$ are subsets of E^n, then by definition [10, p. 241 ff.], $x \in \liminf A_k$ if and only if every neighborhood of x intersects all the A_k with sufficiently high k, and $x \in \limsup A_k$ if and only if every neighborhood of x intersects infinitely many A_k. If $\liminf A_k = \limsup A_k = A$, then we write $A = \lim A_k$, or $A_k \to A$.

THEOREM 5. *If $F_k(t) \to F(t)$ for all t, and all the F_k are Borel-measurable and bounded by the same integrable point-valued function, then $\int F_k \to \int F$.*

Theorem 5 may be restated in terms of continuous set-valued functions. This will be done in Section 5. Neither of the conditions can be omitted in this theorem. The proof of Theorem 5 makes use of two analogues of Fatou's lemma (Propositions 4.1 and 5.1), which are of some interest in their own right.

In the final section, we consider an application to extreme points of sets of vector functions, as treated by Karlin in [12].

Nowhere in this paper do we use the fact that the measure space T is totally finite. In particular, the theorems remain true when T is $(0, \infty)$ or $(-\infty, \infty)$. What *is* essential (to at least some of the theorems) is that T be nonatomic.

PROOF OF THEOREM 2

It will be convenient in this section to replace the euclidean space E^n by an arbitrary separable and complete metric space X. The function F will still be defined on $T = [0, 1]$, but its values will now be subsets of X. A point-valued function f from T to X will be called *Lebesgue-measurable* if $f^{-1}(U)$ is a Lebesgue-measurable subset of T for every open (or equivalently, Borel) subset of X. Recall that an *analytic* subset of X is the continuous image of a Borel subset of X [10, p. 360]; the set-valued function F will be called *analytic* if its graph is an analytic subset of $T \times X$. The following proposition follows from a lemma of von Neumann [11, p. 448, lemma 5]:

PROPOSITION 2.1. *If F is an analytic set-valued function from T to X, then there is a Lebesgue-measurable point-valued function $f: T \to X$ such that $f(t) \in F(t)$ for all t.*

Since every Borel-measurable F is analytic, it follows by setting $X = E^n$ that if F is a Borel-measurable function from T to E^n, then there is an f satisfying the conclusions of Proposition 2.1. If, moreover, F is integrably bounded, then it follows that f is integrable. Then $\int f \in \int F$, and Theorem 2 is proved.

Proof of Theorem 3

The proof is by induction on the dimension n of the space. For $n = 0$ the theorem is immediate, since $F^*(t) = F(t) = \{0\}$ for all t. Suppose the theorem true for dimensions less than n. If the theorem is false in dimension n, then for appropriate F we have $\int F^* \setminus \int F \neq \phi$; let $x \in \int F^* \setminus \int F$. By Theorem 1, $\int F$ is convex, so it has a supporting hyperplane passing through x; that is, there is a vector $a \in E^n$ such that

$$a \cdot y \leqslant a \cdot x \tag{3.1}$$

for all $y \in \int F$.

Since $x \in \int F^*$, it follows that $x = \int f^*$, where $f^*(t) \in F^*(t)$ for all t. Furthermore, f^* may be chosen Borel-measurable; for, every Lebesgue measurable function is equivalent to (i.e., differs on a set of measure 0 from) a Borel measurable function,[2] and according to our convention, "all" means "almost all." Now recall Caratheodory's theorem, which states that if D and D^* are subsets of E^n such that D^* is the convex hull of D, then every point of D^* is a convex combination of $n + 1$ points of D (see Eggleston [13], p. 34 ff.). According to this, for each t there are positive real numbers $\varphi_0(t), \cdots, \varphi_n(t)$ summing to 1, and members $g_0(t), \cdots, g_n(t)$ of $F(t)$, such that

$$f^*(t) = \sum_{j=0}^{n} \varphi_j(t)\, g_j(t). \tag{3.2}$$

We wish to show that the φ_j and the g_j obeying (3.2) can be chosen measurable, and with g_0 integrable. To this end, note that (3.2) implies that for at least one of the g_j, $\sum_{i=1}^n g_j^i(t) \leqslant \sum_{i=1}^n f^{*i}(t)$. Since the indexing of the g_j is of no significance, this means that the subset $G(t)$ of $E^{(n+1)+n(n+1)}$ defined by

$$G(t) = \Big\{(\xi_0, \cdots, \xi_n, x_0, \cdots, x_n) : 0 < \xi_j \leqslant 1 \text{ and } x_j \in F(t) \text{ for all } j,$$

$$\sum_{j=0}^{n} \xi_j = 1, \sum_{i=1}^{n} x_0^i \leqslant \sum_{i=1}^{n} f^{*i}(t), \text{ and } f^*(t) = \sum_{j=0}^{n} \xi_j x_j \Big\}$$

is nonempty for all t. Furthermore, the graph of G is a Borel subset of $E^{(n+1)+n(n+1)+1}$; this follows from the Borel measurability of F and f^*. So by Proposition 2.1, we may choose measurable φ_j and g_j so that

$$(\varphi_0(t), \cdots, \varphi_n(t), g_0(t), \cdots, g_n(t)) \in G(t) \qquad \text{for all} \qquad t.$$

[2] This is immediate for simple functions, and every measurable function is the limit of a sequence of simple functions.

Then the φ_j and the g_j are measurable and obey (3.2). Furthermore, from $g_0(t) \in F(t)$ and the nonnegativity of F it follows that $g_0(t) \geqslant 0$. Hence

$$0 \leqslant g_0^1(t) \leqslant \sum_{i=1}^{n} g_0^i(t) \leqslant \sum_{i=1}^{n} f^{*i}(t),$$

and therefore the integrability of g_0^i follows from that of f^*. Similarly all the g_0^i are integrable, i.e., g_0 is integrable.

We now show that

$$a \cdot g_j(t) \leqslant a \cdot f^*(t) \tag{3.3}$$

for all t and j. Indeed, suppose that

$$a \cdot g_k(t) > a \cdot f^*(t) \tag{3.4}$$

for some k and t, say for $t \in S$, where S has positive measure. For each t, there is a j obeying

$$a \cdot g_j(t) \geqslant a \cdot f^*(t); \tag{3.5}$$

for otherwise, since $\varphi_j(t) > 0$, we have

$$a \cdot f^*(t) = \sum_{j=1}^{n+1} \varphi_j(t)\, a \cdot g_j(t) < \sum_{j=1}^{n+1} \varphi_j(t)\, a \cdot f^*(t) = a \cdot f^*(t),$$

an absurdity. Let us denote the first j that fulfills (3.5) by $j(t)$. Define a function f by

$$f(t) = \begin{cases} g_k(t) & \text{when} \quad t \in S \\ g_{j(t)}(t) & \text{when} \quad t \notin S. \end{cases}$$

Clearly f is measurable, but possibly it is not integrable. For each positive integer m, let $U(m) = \{t : f(t) \leqslant (m, \cdots, m)\}$, and define a sequence of integrable functions f_m by

$$f_m(t) = \begin{cases} f(t) & \text{when} \quad t \in U(m) \\ g_0(t) & \text{when} \quad t \notin U(m). \end{cases}$$

Then $f_m(t) \in F(t)$ for all t, and so (3.1) yields

$$a \cdot \int f_m \leqslant a \cdot \int f^* \tag{3.6}$$

for all f. Now $\bigcup_{m=1}^{\infty} U(m) = T$; therefore

$$\int_{T \setminus U(m)} a \cdot (g_0 - f^*) \to 0. \tag{3.7}$$

Furthermore, for sufficiently large m, $U(m) \cap S$ has positive measure. For such m, then, it follows from (3.4) that

$$\int_{U(m)\cap S} a \cdot f_m = \int_{U(m)\cap S} a \cdot g_k > \int_{U(m)\cap S} a \cdot f^*.$$

In other words, if

$$\epsilon(m) = \int_{S\cap U(m)} a \cdot (f_m - f^*),$$

then $\epsilon(m)$ is monotone increasing in m, and $\epsilon(m) > 0$ for sufficiently large m, say for $m \geqslant m_0$. Now from (3.7) it follows that for m sufficiently large, say $m \geqslant m_1$, we have

$$\int a \cdot (f_m - f^*) = \int_{T\setminus U(m)} a \cdot (g_0 - f^*) \geqslant - \frac{\epsilon(m_0)}{2}.$$

Furthermore, by (3.5) and the definition of f,

$$\int_{U(m)\setminus S} a \cdot f_m = \int_{U(m)\setminus S} a \cdot f \geqslant \int_{U(m)\setminus S} a \cdot f^*.$$

So if $m \geqslant \max(m_1, m_0)$, then

$$a \cdot \int f_m - a \cdot \int f^* = \int a \cdot (f_m - f^*) = \int_{U(m)\cap S} + \int_{U(m)\setminus S} + \int_{T\setminus U(m)}$$

$$\geqslant \epsilon(m_0) + 0 - \frac{\epsilon(m_0)}{2} > 0,$$

contradicting (3.6). This demonstrates (3.3).

Next, we demonstrate

$$a \cdot g_j(t) = a \cdot f^*(t) \tag{3.8}$$

for all t and j. Indeed, suppose that there is a j—say $j = k$—such that for some t, strong inequality holds in (3.3), i.e., $a \cdot g_j(t) < a \cdot f^*(t)$. Applying (3.3) and $\varphi_k(t) > 0$, we deduce

$$a \cdot f^*(t) = \sum_{j=0}^{n} \varphi_j(t)\, a \cdot g_j(t) < \sum_{j=0}^{n} \varphi_j(t)\, a \cdot f^*(t) = a \cdot f^*(t),$$

an absurdity. This proves (3.8).

Let H be the hyperplane $\{y : a \cdot y = 0\}$. Define $E(t) = [F(t) - f^*(t)] \cap H$, and let $E^*(t)$ be the convex hull of $E(t)$. From (3.8) it follows that $g_j(t) - f^*(t) \in E(t)$, and so by (3.2), $0 = f^*(t) - f^*(t) \in E^*(t)$. Since H is of dimension $n - 1$, we may apply the induction hypothesis to E, and deduce

$0 \in \int E$. Let e be such that $e(t) \in E(t)$ for all t and $\int e = 0$. Then for each t, $e(t) + f^*(t) \in F(t)$, and $\int [e + f^*] = \int f^* = x$. Hence $x \in \int F$ after all, contradicting our assumption. This completes the proof of Theorem 3.

To see that Theorem 3 is false if it is not assumed that F is Borel-measurable, let $n = 1$, and let g be the characteristic function of a subset of T with inner measure 0 and outer measure 1. Let $F(t)$ contain the two points $g(t)$ and 2 only. Then $\int F = \{2\}$ but $\int F^* = [1, 2]$.

To see that Theorem 3 is false without the nonnegativity assumption, let $n = 1$ and let $F(t) = \{1/t, -1/t\}$. Then $\int F = \phi$ and $\int F^* = (-\infty, \infty)$.

Proof of Theorem 4

The proof of Theorem 4 is a consequence of the following analogue of Fatou's lemma.

Proposition 4.1. *If $F_1, F_2, \cdots$ is a sequence of set-valued functions that are all bounded by the same integrable point-valued function h, then*

$$\int \limsup F_k \supset \limsup \int F_k .$$

Proof. Suppose $x \in \limsup \int F_k$. Then x is a limit point of a sequence $\int f_k$, where $f_k(t) \in F_k(t)$ for each k and t; that is, there is a subsequence of $\int f_k$ converging to x. We wish to show that $x \in \int \limsup F_k$; for this purpose we may assume without loss of generality that x is actually the limit of the $\int f_k$, i.e. that the subsequence converging to x is the whole original sequence.

The f_k can be considered real-valued functions on $\{1, \cdots, n\} \times T$; since they are integrable, it follows that they are members of the Banach space $L^1 = L^1(\{1, \cdots, n\} \times T)$. Because the f_k are all bounded by the integrable function h, it follows that there is a subsequence with a weak limit, which we call f (see Dunford-Schwartz [14], Theorem IV.8.9, p. 292). Again, we may assume without loss of generality that f_k actually converges to f weakly.

Let $\{\epsilon_1, \epsilon_2, \cdots\}$ be a sequence of positive real numbers tending to 0. Since $\{f_1, f_2, \cdots\}$ approaches f weakly, it follows that also $\{f_m, f_{m+1}, \cdots\}$ approaches f weakly, for all m. Hence from a known theorem, it follows that there is a sequence of convex combinations of $f_m, f_{m+1}, \cdots$ that approaches f in the norm of L^1 ([14] Corollary V. 3.14, p. 422). For each m, let g_m be such a convex combination with $\| g_m - f \| \leqslant \epsilon_m$, where $\| \; \|$ denotes the L^1 norm. Then $g_m \to f$ in the norm of L^1. Hence by a known theorem, there is a subsequence of the g_m that converges to f almost everywhere (see [14], Theorem III, 3.6(i), p. 122, for a proof that norm convergence implies convergence in measure, and Corollary III, 6.13(a), p. 150, for a proof that convergence in measure implies convergence a.e. of a subsequence); without loss of generality[3] let it be the whole original sequence. So for all $t, g_m(t) \to f(t)$, and $g_m(t)$ is a convex combination of $\{f_m(t), f_{m+1}(t), \cdots\}$. Since the latter are points of E^n, it follows from Caratheodory's theorem [13, pp. 34 ff.] that

$$g_m(t) = \sum_{j=0}^{n} \theta_{jm}(t)\, e_{jm}(t),$$

where the $\theta_j(t)$ are nonnegative and sum to 1, and $e_{0m}(t), \cdots, e_{nm}(t)$ are chosen from among $f_m(t), f_{m+1}(t), \cdots$. Now for each t, we can choose a subsequence of the $g_m(t)$ such that all the corresponding subsequences of $\{\theta_{0m}(t)\}, \cdots, \{\theta_{nm}(t)\}, \{e_{0m}(t)\}, \cdots$, and $\{e_{nm}(t)\}$ converge. The limits of the θ's in these subsequences must be nonnegative numbers summing to 1, and the limits of the e's must be limit points of the $f_m(t)$. Hence for each t,

$$f(t) = \lim g_m(t) = \sum_{j=0}^{n} \theta_j e_j(t),$$

where the θ_j are nonnegative and sum to 1, and the $e_j(t)$ are limit points of $\{f_1(t), f_2(t), \cdots\}$. So if we let $G(t)$ be the set of limit points of $\{f_k(t)\}$, and $G^*(t)$ the convex hull of $G(t)$, then we have shown that $f(t) \in G^*(t)$ for each t. Hence $\int f \in \int G^*$.

As we have noted above, every Lebesgue-measurable function is equivalent to a Borel-measurable function; so we may assume that the f_k are Borel-measurable. Then it follows easily that G is Borel-measurable. So we may apply Theorem 3 and deduce that $\int G^* = \int G$. Hence $\int f \in \int G$. But since $f_k(t) \in F_k(t)$, it follows that every limit point of the $F_k(t)$ will be a member of $\limsup F_k(t)$, i.e., $G(t) \subset \limsup F_k(t)$. Hence $\int f \in \int \limsup F_k$. But from the weak convergence of f_k to f it follows that $\int f = \lim \int f_k = x$, so that $x \in \int \limsup F_k$. This completes the proof of Proposition 4.1.

Suppose now that F is closed and integrably bounded; set $F_1 = F_2 = \cdots = F$. Then $\limsup F_k = \mathrm{cl}\, F = F$ and $\limsup \int F_k = \mathrm{cl} \int F$, where cl denotes "closure" (see [10], p. 243, IV.6). So by Proposition 4.1,

$$\int F = \int \limsup F_k \supset \limsup \int F_k = \mathrm{cl} \int F,$$

and hence $\int F$ is closed. Since it is bounded by the integral of the function h that bounds F, it follows that it is compact, proving Theorem 4.

It is possible to prove Theorem 4 somewhat more simply by a direct application of [14, V.3.14]; but Proposition 4.1 is interesting for its own sake, and is needed in the proof of Theorem 5.

[3] This involves cutting down the original sequence of f_k's and using the corresponding subsequence of the g_k's.

If we replace the integrable boundedness condition in Theorem 4 by the assumption that F is nonnegative, then $\int F$ need not even be closed. For a counterexample, let $n = 2$, let $g(t) = ((1 - t)/t, t/(1 - t))$, and let $F(t) = \{0, g(t)\}$. Then F is the union of the open positive quadrant with the origin $\{0\}$.

PROOF OF THEOREM 5

We first prove another analogue of Fatou's lemma, as follows:

PROPOSITION 5.1. *If all the F_k are Borel-measurable and bounded by the same integrable point-valued function, then*

$$\int \liminf F_k \subset \liminf \int F_k .$$

PROOF. Let $x \in \int \liminf F_k$. Then $x = \int f$, where $f(t) \in \liminf F_k(t)$ for each t. Since f is equivalent to a Borel-measurable function, it may be assumed to be Borel-measurable. Now the space $E^n \times E^n \times \cdots$ may be metrized so that it is complete and so that its topology is the usual product topology [10, p. 313]. For each t, define a subset $G(t)$ of $E^n \times E^n \times \cdots$ by

$$G(t) = \{(x_1, x_2, \cdots) : x_1 \in F_1(t), x_2 \in F_2(t), \cdots, \text{and } \lim x_k = f(t)\}.$$

Then the statement $f(t) \in \liminf F_k(t)$ is precisely equivalent to the statement $G(t) \neq \phi$ [10, p. 242], and hence $G(t) \neq \phi$ for each t. Furthermore, G is easily seen to be Borel-measurable and hence analytic. So by Proposition 2.1, there is a measurable function g from T to $E^n \times E^n \times \cdots$ such that $g(t) \in G(t)$ for each t; that is, a sequence $f_1, f_2, \cdots$ of measurable functions from T to E^n, such that $f_k(t) \in F_k(t)$ for each t, and $\lim f_k(t) = f(t)$. Now since $f_k(t) \in F_k(t)$, all the f_k are bounded by the same integrable point valued function. Hence from Lebesgue's bounded convergence theorem, it follows that $\int f_k \rightarrow \int f = x$. But $\int f_k \in \int F_k$, and so $x \in \liminf \int F_k$ [10, p. 242]. This completes the proof of Proposition 5.1.

If $F(t) = \liminf F_k(t) = \limsup F_k(t)$ for all t, then by Propositions 4.1 and 5.1,

$$\int F = \int \liminf F_k \subset \liminf \int F_k \subset \limsup \int F_k \subset \int \limsup F_k = \int F;$$

hence equality holds throughout, and so $\lim \int F_k$ exists and equals $\int \lim F_k$. This proves Theorem 5.

Theorem 5 is false without the Borel-measurability assumption. Indeed, let $n = 1$, let g be the characteristic function of a subset of T with inner measure

0 and outer measure 1, and let $F_k(t) = \{g(t)/k\}$. Then $\lim \int F_k = \phi$ but $\int \lim F_k = \{0\}$.

Let A be an arbitrary subset of a metric space X, and let G_x be a set-valued function defined for x in A, whose values are subsets of E^n. G is called *upper-semicontinuous* if $x_n \to x$ implies $G_x \supset \limsup G_{x_n}$ for all x_n and x in A; *lower-semicontinuous* if $x_n \to x$ implies $G_x \subset \liminf G_{x_n}$ for all x_n and x in A; and *continuous* if $x_n \to x$ implies $G_x = \lim G_{x_n}$ for all x_n and x in A. This is the same as the standard definition of these terms (see, for example, Karlin [15], p. 409).

COROLLARY 5.2. *Let $F_x(t)$ be a set-valued function defined for $t \in T$ and $x \in A$, all of whose values are bounded by the same integrable point-valued function, and such that F_x is Borel-measurable for each fixed $x \in A$. Then if $F_x(t)$ is upper-semicontinuous in x for each fixed t, then $\int F_x$ is upper-semicontinuous; if $F_x(t)$ is lower-semicontinuous in x for each fixed t, then $\int F_x$ is lower semi-continuous; and if $F_x(t)$ is continuous for each fixed t, then $\int F_x$ is continuous.*

PROOF. Follows from Propositions 4.1 and 5.1.

APPLICATIONS TO EXTREME POINTS OF SETS OF VECTOR FUNCTIONS

In this section, we make use of Proposition 2.1 only; the other results will not be used. Let A be a compact convex subset of E^n, B the set of its extreme points, $\mathrm{cl}\,(B)$ the closure of B. If $n \geqslant 3$, it is not necessarily true that $\mathrm{cl}\,(B) = B$. Let $\mathscr{M}_A$, $\mathscr{M}_B$, and $\mathscr{M}_{\mathrm{cl}\,(B)}$ be the sets of all measurable functions from T to A, B, and $\mathrm{cl}\,(B)$ respectively. Since the space of all measurable functions from T to E^n has a linear structure, we may discuss the extreme points of $\mathscr{M}_A$. In [12], Karlin proved that the set of extreme points of $\mathscr{M}_A$ includes $\mathscr{M}_B$ and is included in $\mathscr{M}_{\mathrm{cl}\,(B)}$.

PROPOSITION 6.1. *The set of extreme points of $\mathscr{M}_A$ is precisely $\mathscr{M}_B$.*

PROOF. Clearly, every point of $\mathscr{M}_B$ is an extreme point of $\mathscr{M}_A$. Conversely, let f be an extreme point of $\mathscr{M}_A$. If f is not in $\mathscr{M}_B$, then for some t, $f(t)$ is not an extreme point of A. Hence for each t we may choose $g(t)$ and $h(t)$ in A so that $f(t) = \frac{1}{2}g(t) + \frac{1}{2}h(t)$, and $g(t)$ and $h(t)$ differ from $f(t)$ for at least some t. Because of Proposition 2.1, g and h may be chosen so as to be measurable as well. Then g and h are in $\mathscr{M}_A$ and $f = \frac{1}{2}g + \frac{1}{2}h$, contradicting the extreme point property of f. This proves the proposition.

Another situation treated by Karlin in [12] is the following: Let $\mu_1, \cdots, \mu_m$ be a set of nonatomic totally finite measures on T, and let $a_1, \cdots, a_m$, $b_1, \cdots, b_m$ be in E^n. Let A and B be as above. Let $\mathscr{G}$ be the subset of $\mathscr{M}_A$

consisting of functions f such that $b_i \leqslant \int f d\mu_i \leqslant a_i$ for $i = 1, \cdots, m$. Karlin proved that the extreme points of $\mathscr{G}$ are contained in $\mathscr{M}_{\mathrm{cl}\,(B)}$.

PROPOSITION 6.2. *The extreme points of $\mathscr{G}$ are contained in $\mathscr{M}_B$.*

PROOF. The proof follows Karlin's ideas, but use of Proposition 2.1 makes it simpler and yields the stronger result. Let f be an extreme point of $\mathscr{G}$, and suppose it is not in $\mathscr{M}_B$. Construct g and h as in the previous proof, and let $e = \frac{1}{2} g - \frac{1}{2} h$, so that $f + e = g \in \mathscr{M}_A$ and $f - e = h \in \mathscr{M}_A$. For $S \subset T$, set $\mu(S) = \{\int_S e d\mu_1, \cdots, \int_S e d\mu_m\}$. Then μ is a vector measure of dimension nm. Applying Lyapunov's theorem on vector measures [16], we obtain a subset S of T such that $\mu(S) = \frac{1}{2}\mu(T)$. Define e' on T by $e'(t) = e(t)$ for $t \in S$, and $e'(t) = - e(t)$ for $t \notin S$. Now define f_1 and f_2 by $f_1 = f + e'$, $f_2 = f - e'$. Then $f = \frac{1}{2} f_1 + \frac{1}{2} f_2$, f_1 and f_2 are in $\mathscr{M}_A$, and

$$\int f_1 d\mu_i = \int f d\mu_i + \int_S e d\mu_i - \int_{T \setminus S} e d\mu_i$$

$$= \int f d\mu_i + \tfrac{1}{2} \int e d\mu_i - \tfrac{1}{2} \int e d\mu_i = \int f d\mu_i .$$

Hence $f_1 \in \mathscr{G}$. Similarly $f_2 \in \mathscr{G}$, and the proof is complete.

If F is the set-valued function defined by $F(t) = A$ for all t, then $\mathscr{F} = \mathscr{M}_A$. Propositions analogous to 6.1 and 6.2 can be proved in the general case, when F need not be constant, but is assumed to have compact convex values. In this connection, we note that Theorems 3 and 4 of this paper are generalizations of Theorem 3 and Corollary 1 of [12].

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